THE DYNAMIC PROBLEMS OF THERMAL ELASTICITY

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We examine the stresses which arise in an elastic half-space when the surface of the latter is suddenly heated, bearing in mind the rates of expansion-wave propagation in an elastic medium and the finite rate of heat propagation.

The dynamic problems of thermal elasticity were first treated in [1, 2], these papers devoted to the study of the stressed state resulting from the sudden heating of the plane surrounding an elastic half-space. These problems were subsequently again considered by Mura [3, 4] who apparently was unaware of [1, 2]. Further investigation of the dynamic problems of thermal elasticity are described in [5-9] and elsewhere. In each of these references the temperature field is described with the Fourier equation.

It is demonstrated in [10, 11] that for metals with a great temperature gradient there exists no classical relationship between the heat flow and the gradient. This indicates that the solutions of the heat-conduction equation used in [1-9] do not correspond to the true temperature field. For the solution of the dynamic problems of thermal elasticity we therefore have to employ the new hyperbolic heat-conduction equation derived with consideration of the finite rate of heat propagation [12-15].

Let us consider an elastic half-space at a temperature T_0; the temperature field [15] is described by the equation

$$\frac{\partial T(z,\ t)}{\partial t}+\frac{a}{c_q^2}\,\frac{\partial^2 T(z,\ t)}{\partial t^2}=a\,\frac{\partial^2 T(z,\ t)}{\partial z^2}. \tag{1}$$

At the instant $t = 0$ the surface temperature of the half-space varies jumpwise to T_c and then remains constant. To determine the temperature field we should solve Eq. (1) with the boundary conditions

$$T(z,\ 0)=T_0,\quad \frac{\partial T(z,\ 0)}{\partial t}=0,\quad T(0,\ t)=T_c. \tag{2}$$

To determine the stresses [16] we have to solve the equation

$$\frac{\partial^2\sigma_{zz}(z,\ t)}{\partial z^2}-\frac{1}{c^2}\,\frac{\partial^2\sigma_{zz}(z,\ t)}{\partial t^2}=\frac{1+\mu}{1-\mu}\,\rho\alpha\,\frac{\partial^2 T(z,\ t)}{\partial t^2}. \tag{3}$$

We will assume that the half-space is initially free of stresses and that there are no stresses on its surface as the body is heated. The boundary conditions for Eq. (3) will then be

$$\sigma_{zz}(z,\ 0)=0,\quad \frac{\partial\sigma_{zz}(z,\ 0)}{\partial t}=0,\quad \sigma_{zz}(0,\ t)=0. \tag{4}$$

After determining σ_{zz} and T we can determine the remaining components of the stress tensor easily from [16]:

$$\sigma_{xx}=\sigma_{yy}=\frac{1}{1-\mu}\{\mu\sigma_{zz}-E\alpha(T-T_0)\}. \tag{5}$$

Let us introduce the following dimensionless quantities:

$$\sigma=\frac{\sigma_{zz}(1-2\mu)}{E\alpha(T_c-T_0)},\quad \theta=\frac{T-T_0}{T_c-T_0},\quad \xi=\frac{cz}{a},\quad \tau=\frac{c^2 t}{a},\quad \mathrm{M}=\frac{c}{c_q}. \tag{6}$$

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With the aid of (6) we can write Eqs. (1)-(4) in the form

$$\frac{\partial \theta(\xi, \tau)}{\partial \tau} + M^2 \frac{\partial^2 \theta(\xi, \tau)}{\partial \tau^2} = \frac{\partial^2 \theta(\xi, \tau)}{\partial \xi^2}, \tag{7}$$

$$\theta(\xi, 0) = 0, \quad \frac{\partial \theta(\xi, 0)}{\partial \tau} = 0, \quad \theta(0, \tau) = 1, \tag{8}$$

$$\frac{\partial^2 \sigma(\xi, \tau)}{\partial \xi^2} - \frac{\partial^2 \sigma(\xi, \tau)}{\partial \tau^2} = \frac{\partial^2 \theta(\xi, \tau)}{\partial \tau^2}, \tag{9}$$

$$\sigma(\xi, 0) = 0, \quad \frac{\partial \sigma(\xi, 0)}{\partial \tau} = 0, \quad \sigma(0, \tau) = 0. \tag{10}$$

After applying the Laplace transform

$$\bar{\theta}(\xi, s) = \int_0^\infty \theta(\xi, \tau) e^{-s\tau} d\tau$$

to Eq. (7), with consideration of the boundary conditions (8) and of the fact that as $\xi \to \infty$ the temperature must remain bounded, we find the image of the temperature, i.e.,

$$\bar{\theta}(\xi, s) = \frac{1}{s} \exp\left[-\xi \sqrt{s(1 + M^2 s)}\right]. \tag{11}$$

The reconversion leads to the following relationship:

$$\theta(\xi, \tau) = \eta(\tau - \xi M) \exp\left(-\frac{\xi}{2M}\right) + \frac{\xi}{2M} \int_{\frac{\xi}{2M}}^{\frac{\tau}{2M^2}} \exp(-x) \frac{I_1\left[\sqrt{x^2 - \left(\frac{\xi}{2M}\right)^2}\right]}{\sqrt{x^2 - \left(\frac{\xi}{2M}\right)^2}} dx, \tag{12}$$

where

$$\eta(\tau - \xi M) = 0 \quad \text{for } \tau < \xi M,$$
$$\eta(\tau - \xi M) = 1 \quad \text{for } \tau > \xi M.$$

After application of the Laplace transform

$$\bar{\sigma}(\xi, s) = \int_0^\infty \sigma(\xi, \tau) e^{-s\tau} d\tau$$

to Eq. (9), with consideration of the first two conditions in (10), we have

$$\frac{d^2 \bar{\sigma}(\xi, s)}{d\xi^2} - s^2 \bar{\sigma}(\xi, s) = s^2 \bar{\theta}(\xi, s). \tag{13}$$

Having substituted (11) into Eq. (13) and bearing in mind that $\bar{\sigma}(0, s) = 0$ and that as $\xi \to \infty$ the stresses must remain bounded, we find for the images

$$\bar{\sigma}(\xi, s) = \frac{1}{s(1 - M^2) - 1} \left[\exp(-\xi s) - \exp\left[-\xi \sqrt{s(1 + M^2 s)}\right]\right]. \tag{14}$$

Reconversion yields

$$\sigma(\xi, \tau) = \frac{\exp\left(\frac{\tau - \xi}{1 - M^2}\right)}{1 - M^2} \left\{\eta(\tau - \xi) - \eta(\tau - \xi M) \exp\left(\frac{\xi}{2M} \frac{M - 1}{M + 1}\right)\right\}$$

$$- \frac{1}{1 - M^2} \frac{\xi}{2M} \int_{\xi/2M}^{\tau/2M^2} \exp\left[-x \frac{1 + M^2}{1 - M^2}\right] \frac{I_1\left[\sqrt{x^2 - \left(\frac{\xi}{2M}\right)^2}\right]}{\sqrt{x^2 - \left(\frac{\xi}{2M}\right)^2}} dx. \tag{15}$$

If the heat-propagation rate c_q is considerably greater than the rate of propagation for the elastic waves, we have M = 0 and the solution coincides with the data in [1] and [4]:

$$\sigma(\xi, \tau) = \eta(\tau - \xi) \exp(\tau - \xi) - \frac{1}{2} e^{\tau} \left[e^{\xi} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} + \sqrt{\tau} \right) + e^{-\xi} \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\tau}} - \sqrt{\tau} \right) \right]. \tag{16}$$

NOTATION

T is the temperature;
z is a coordinate;
t is the time;
a is the thermal diffusivity;
c_q is the rate of heat propagation;
σ_{zz} is the stress;
c is the rate of propagation for the elastic waves;
μ is the Poisson ratio;
α is the coefficient of thermal expansion;
ρ is the mass of the volume unit;
E is Young's modulus.

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